

SPECTRAL EQUALITY FOR C_0 SEMIGROUPS

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ABSTRACT. In this paper, we give conditions for which the C_0 semigroups satisfies spectral equality for semiregular, essentially semiregular and semi-Fredholm spectrum. Also, we establish the spectral inclusion for B-Fredholm spectrum of a C_0 semigroups.

1. INTRODUCTION AND PRELIMINARIES.

Let X a Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operator on X . For $T \in \mathcal{B}(X)$, let $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{n \geq 0} R(T^n)$, denote respectively the kernel, the range and the hyper-range of T .

We denote by $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_r(T)$ and $\sigma_{su}(T)$ respectively the resolvent set, the spectrum, the point spectrum, the approximate point spectrum, the residual spectrum and the surjective spectrum of T .

For $\lambda \in \rho(T)$, we denote by $R(\lambda, T) = (\lambda - T)^{-1} \in \mathcal{B}(X)$ the resolvent operator of T .

Recall that T is semiregular if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$ and T is essentially semiregular if $R(T)$ is closed and there exists a finite dimensional subspace $G \subseteq X$ such that $N(T) \subseteq R^\infty(T) + G$. A closed linear operator T is said to be upper semi-Fredholm if $R(T)$ is closed and $\dim N(T) < \infty$, and T is the lower semi-Fredholm if $\text{codim} R(T) < \infty$, and if the $\dim N(T)$ and $\text{codim} R(T)$ are both finite then T is called a Fredholm operator.

Let $A \in \mathcal{B}(X)$ and n a nonnegative integer. Define T_n to be the restriction of T to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. If for some integer n the range space $R(T^n)$ is closed and the induced operator $T_n \in \mathcal{B}(R(T^n))$ is Fredholm, then T will be said B-Fredholm. In a similar way, if T_n is upper semi-Fredholm (respectively, lower semi-Fredholm) operator, then T is called upper semi B-Fredholm (respectively, lower semi B-Fredholm), see [2] for more detail about semi B-Fredholm operators.

This classes of operators lead to the definition of the semiregular spectrum, essentially semiregular spectrum, semi-Fredholm spectrum, Fredholm spectrum, upper semi-B-Fredholm spectrum, lower semi-B-Fredholm spectrum and the B-Fredholm spectrum defined by:

$$\begin{aligned}\sigma_\gamma(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semiregular}\}; \\ \sigma_{\gamma e}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not essentially semiregular}\}; \\ \sigma_\pi(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi-Fredholm}\};\end{aligned}$$

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$$\begin{aligned}
\sigma_F(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm}\}; \\
\sigma_{uBF}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-B-Fredholm}\}; \\
\sigma_{lBF}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-B-Fredholm}\}; \\
\sigma_{BF}(T) &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Fredholm}\}.
\end{aligned}$$

Let X^* denote the dual space of X and A^* the adjoint operator of A with domain $D(A)$. Define the reduced minimum modulus $\gamma(A)$ by setting

$$\gamma(A) = \inf \left\{ \frac{\|Au\|}{d(u, N(A))}, u \in D(A) \setminus N(A) \right\}.$$

It is known that if $D(A)$ is dense in X , then $\gamma(A) = \gamma(A^*)$ and $\gamma(A) > 0$ if and only if $R(A)$ is closed.

Recall that the family $(T(t))_{t \geq 0}$ of operators on X is called a strongly continuous semigroup of operators (in short C_0 -semigroups) if:

- (1) $T(0) = I$;
- (2) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;
- (3) $\lim_{t \downarrow 0} T(t)x = x$ for every $x \in X$.

Let $\mathcal{T} = (T(t))_{t \geq 0}$ be a strongly continuous semigroup on X . The linear operator A defined in

$$D(A) = \{x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exist}\}$$

by

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} = \frac{d^+ T(t)x}{dt} \Big|_{t=0} \text{ for } x \in D(A)$$

is the infinitesimal generator of the semigroup $T(t)$ and $D(A)$ its domain.

Let $\mathcal{T} = (T(t))_{t \geq 0}$ be a strongly continuous semigroup with generator A on X . We will denote the type (growth bound) of τ by

$$\begin{aligned}
\omega_0 &= \lim_{t \rightarrow \infty} \frac{\ln \|T'(t)\|}{t} \\
&= \inf \{ \omega \in \mathbb{R} : \text{there exists } M \text{ such that } \|T(t)\| \leq Me^{\omega t}, t \geq 0 \}.
\end{aligned}$$

Let $A \in \mathcal{B}(X)$. The quasi-nilpotent part and the analytic part play an important role in local spectral theory, see the monographic of Laursen and Neumann [9]. Recall that the quasi nilpotent part of A denoted $H_0(A)$ is given by

$$H_0(A) = \left\{ x \in \bigcap_{n \geq 0} D(A^n) : \lim_{n \rightarrow \infty} \|A^n x\|^{\frac{1}{n}} = 0 \right\}.$$

Also, the algebraic core and the analytic core noted respectively by $C(T)$, $K(T)$ are defined by:

$$\begin{aligned}
C(T) &= \{x \in X : \exists (x_n)_{n \geq 0} \subset X, x_0 = x \text{ and } Tx_n = x_{n-1} \forall n \geq 1\} \\
K(T) &= \{x \in X : \exists (x_n)_{n \geq 0} \subset X \text{ and } \delta > 0 \text{ such that } x_0 = x \text{ and} \\
&\quad Tx_n = x_{n-1} \forall n \geq 1 \text{ and } \|x_n\| \leq \delta^n \|x\|\}.
\end{aligned}$$

In [15, Lemma 2.3] it is proved that if $A, B \in \mathcal{B}(X)$ are such that $AB = BA$, then :

$$1. K(AB) \subseteq K(A) \bigcap K(B).$$

$$2.C(AB) \subseteq C(T) \bigcap C(B).$$

Recall that some spectral inclusions for various reduced spectra are studied in [6] and [3]. The authors proved that

$$e^{t\nu(A)} \subseteq \nu(T(t)) \subseteq e^{t\nu(A)} \bigcup \{0\},$$

where $\nu(\cdot) \in \{\sigma_p(\cdot), \sigma_{ap}(\cdot), \sigma_r(\cdot)\}$, point spectrum, approximative spectrum and residual spectrum.

In section two of this work, after given the relationship between the analytical core (respectively, the algebraic core) of a strongly continuous semigroups with its infinitesimal generator, we will prove the spectral equality for a C_0 semigroup for semiregular, essentially semiregular and semi-Fredholm spectrum, respectively. In section 3, we will prove the spectral inclusion for the B-Fredholm spectrum.

2. SINGULAR SPECTRUM FOR SEMIGROUPS GENERATORS

Let $T(t)$ be a C_0 -semigroup on a Banach space X and let A be its infinitesimal generator. In this section we will study the relations between the spectrum of A and the spectrum of each one of the operators $T(t)$, $t \geq 0$. For this we will begin with the following lemmas proved in [3, 6, 12]. It will be needed in the sequel.

Lemma 2.1. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$. Then, for every $\lambda \in \mathbb{C}$ and $t > 0$, the following identities hold:*

(1)

$$\begin{aligned} (e^{\lambda t} - T(t))x &= (\lambda - A) \int_0^t e^{\lambda(t-s)} T(s)x ds, \quad \lambda \in \mathbb{C}, x \in X \\ &= \int_0^t e^{\lambda(t-s)} T(s)(\lambda - A)x ds, \quad \lambda \in \mathbb{C}, x \in D(A). \end{aligned}$$

(2) $R(e^{\lambda t} - T(t)) \subseteq R(\lambda - A)$.

(3) $N(\lambda - A) = \bigcap_{t \geq 0} N(e^{\lambda t} - T(t)) \subseteq N(e^{\lambda t} - T(t))$.

(4) $N(e^{\lambda t} - T(t)) = \overline{\text{span}} \bigcup_{k \in \mathbb{Z}} N(\lambda + \frac{2\pi i k}{t} - A)$.

Lemma 2.2. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ and $B(\lambda, t) = \int_0^t e^{\lambda(t-s)} T(s)x ds$. Then for every $\lambda \in \mathbb{C}$, $t > 0$ and $n \in \mathbb{N}$, we have the following:*

- (1) $(e^{\lambda t} - T(t))^n(x) = (\lambda - A)^n B(\lambda, t)^n(x)$, $\lambda \in \mathbb{C}, x \in X$
 $= B(\lambda, t)^n (\lambda - A)^n(x)$, $\lambda \in \mathbb{C}, x \in D(A)$;
- (2) $R(e^{\lambda t} - T(t))^n \subseteq R(\lambda - A)^n$;
- (3) $N(\lambda - A)^n \subseteq N(e^{\lambda t} - T(t))^n$;
- (4) $H_0(\lambda - A) \subseteq H_0(e^{\lambda t} - T(t))$.

In this direction; we prove the following lemma.

Lemma 2.3. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ and $B(\lambda, t) = \int_0^t e^{\lambda(t-s)} T(s)x ds$. Then, for every $\lambda \in \mathbb{C}$, $t > 0$ and $n \in \mathbb{N}$, we have the following:*

- (1) $K(e^{\lambda t} - T(t)) \subseteq K(\lambda - A)$;
- (2) $C(e^{\lambda t} - T(t)) \subseteq C(\lambda - A)$.

Proof: (1) let $x \in K(e^{\lambda t} - T(t))$, then there exists $(x_n)_{n \geq 0} \subset X$ and $\delta > 0$ such that $x_0 = x, (e^{\lambda t} - T(t))x_n x_{n-1}$ and $\|x_n\| \leq \delta^n \|x\|$.

Let $(y_n)_{n \geq 0}$ be defined by

$$y_n = B^n(\lambda, t)x_n,$$

then $y_0 = x_0 = x$, hence $(\lambda - A)y_n = (\lambda - A)B^n(\lambda, t)x_n = (\lambda - A)B(\lambda, t)B^{n-1}(\lambda, t)x_n$ as $B^n(\lambda, t)x \in D(A^n) \subseteq D(A)$, then $B^n(\lambda, t)$ and $(\lambda - A)$ commutes for all n and from (1), we have $(\lambda - A)y_n = B^{n-1}(\lambda, t)(\lambda - A)B(\lambda, t)x_n = B^{n-1}(\lambda, t)(e^{\lambda t} - T(t))x_n = B^{n-1}(\lambda, t)x_{n-1} = y_{n-1}$ and

$$\|y_n\| = \|B^n(\lambda, t)x_n\| \leq \|B^n(\lambda, t)\|\delta^n \|x\|.$$

Let $\delta' = \|B(\lambda, t)\|\delta$ it follows that $\|y_n\| \leq \delta'^n \|x\| \Rightarrow x \in K(\lambda - A)$.

(2) Resulting directly from the (5).

In [4, 5, 12] the authors studied and developed a spectral theory for semi groups and their generators. Precisely, they proved the following two theorems:

Theorem 2.1. *For the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$, we have the following spectral inclusion*

$$e^{t\nu(A)} \subseteq \nu(T(t)) \setminus \{0\}; \forall t \geq 0,$$

for $\nu(\cdot) \in \{\sigma_\gamma(\cdot); \sigma_{\gamma_e}(\cdot); \sigma_\pi(\cdot); \sigma_F(\cdot)\}$

and

Theorem 2.2. [12] *Let $T(t)$ be a C_0 semigroup and let A be its infinitesimal generator. Then*

$$e^{t\sigma_{ap}(A)} \subset \sigma_{ap}(T(t)) \subset e^{t\sigma_{ap}(A)} \bigcup \{0\}.$$

More precisely, if $\lambda \in \sigma_{ap}(A)$, then $e^{\lambda t} \in \sigma_{ap}(T(t))$ and if $e^{\lambda t} \in \sigma_{ap}(T(t))$ there exists $k \in \mathbb{Z}$ such that $\lambda_k = \lambda + \frac{2\pi i k}{t} \in \sigma_{ap}(A)$.

In [12] it is proved for a C_0 -semigroup $T(t)$ with A its infinitesimal generator that, if $T(t)$ is differentiable for $t > t_0$ and $\lambda \in \sigma(A)$, then $\lambda e^{\lambda t} \in \sigma(AT(t))$. In the following lemma, we will prove that this result also holds for the point spectrum.

Lemma 2.4. *Let $T(t)$ be a C_0 -semigroup and let A be its infinitesimal generator. If $T(t)$ is differentiable for $t > t_0$ and $\lambda \in \sigma_p(A)$, then $\lambda e^{\lambda t} \in \sigma_p(AT(t))$.*

Proof: If $\lambda \in \sigma_p(A)$, $t > t_0$, $\exists x \in D(A), x \neq 0$ such that $(\lambda I - A)x = 0$. Assuming now that $t > t_0$ and differentiating the equality 1. of lemma 2.1 with respect to t , we obtain:

$$\lambda e^{\lambda t} x - AT(t)x = (\lambda I - A)B'(\lambda, t)x \text{ for every } x \in X.$$

$$= B'(\lambda, t)(\lambda I - A)x \text{ for every } x \in D(A).$$

This implies that $\lambda e^{\lambda t} x - (AT(t))x = 0$, that is $\lambda e^{\lambda t} \in \sigma_p(AT(t))$.

To continue the development of a spectral theory for semi groups and their generators, we prove that the formula holds for semiregular spectrum, essentially semiregular spectrum, Fredholm spectrum and semi Fredholm spectrum. For this we begin with the following result proved in [11] which will be used to prove the following theorem:

Proposition 2.1. (1) If $A \in \mathcal{B}(X)$ is semiregular, then the operator $\widehat{A} : X/R^\infty(A) \rightarrow X/R^\infty(A)$ induced by A is bounded below.

(2) If $A \in \mathcal{B}(X)$ is essentially semiregular, then the operator $\widehat{A} : X/R^\infty(A) \rightarrow X/R^\infty(A)$ induced by A is upper semi-Fredholm.

Theorem 2.3. For the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$, we have

$$e^{t\nu(A)} \subset \nu(T(t)) \subset e^{t\nu(A)} \bigcup \{0\}; \text{ for all } t \geq 0.$$

More precisely, if $\lambda \in \nu(A)$, then $e^{\lambda t} \in \nu(T(t))$ and if $e^{\lambda t} \in \nu(T(t))$ and $(T(t))_{t \geq 0}$ is a periodic with period t , then there exists $k \in \mathbb{Z}$ such that $\lambda_k = \lambda + \frac{2\pi i k}{t} \in \nu(A)$. where $\nu \in \{\sigma_\gamma; \sigma_{\gamma e}; \sigma_\pi; \sigma_F\}$

Proof: if $\lambda \in \nu(A)$ then $e^{\lambda t} \in \nu(T(t))$ see [4, theorem 2.1] and [14, theorem 2].

To prove the second inclusion as:

For the regular spectrum: Let $t_0 > 0$ be fixed and suppose that for all $\lambda \in \mathbb{C} \setminus \lambda + 2\pi i k t_0^{-1}$ such that $(\lambda - A)$ is semiregular. We show that $(e^{\lambda t_0} - T(t_0))$ is semiregular. We consider the closed $(T(t))_{t \geq 0}$ -invariant subspace $M = R^\infty(e^{\lambda t_0} - T(t_0))$ of X and quotient semigroup, $(\widehat{T}(t))_{t \geq 0}$ defined on X/M by: $\widehat{T}(t)\widehat{x} = \widehat{T(t)x}$, for $\widehat{x} \in X/M$, with generator \widehat{A} defined by:

$$D(A) = \{\widehat{x}, x \in D(A)\} \text{ and } A\widehat{x} = \widehat{Ax}, \text{ for all } \widehat{x} \in D(A).$$

From (1) of proposition(2.1) it flows that the operator $(\lambda - \widehat{A})$ is bounded below for all $\lambda \in \mathbb{C} \setminus \lambda + 2\pi i k t_0^{-1}$ and for all $k \in \mathbb{Z}$. Thus, $\lambda \notin \sigma_{ap}(\widehat{A})$. By virtue of Theorem (2.2), we get $e^{\lambda t_0} \notin \sigma_{ap}(\widehat{T}(t_0))$ in consequence, the operator $(e^{\lambda t_0} - \widehat{T}(t_0))$ is bounded below. If $(e^{\lambda t_0} - T(t_0))x = 0$ then $(e^{\lambda t_0} - \widehat{T}(t_0))(x + M) = 0$ and the injective of $(e^{\lambda t_0} - \widehat{T}(t_0)) \Rightarrow x \in M$. thus $N(e^{\lambda t_0} - T(t_0)) \subset M$.

We found that

$$N(e^{\lambda t_0} - T(t_0)) \subset R^\infty(e^{\lambda t_0} - T(t_0))$$

Now, let us show $R(e^{\lambda t_0} - T(t_0))$ is closed. To do this, let a sequence $(y_n)_n$ of elements of $R(e^{\lambda t_0} - T(t_0))$ and $y_n \rightarrow y$, then there exists a sequence $(v_n)_n \in X$ such that

$$R(e^{\lambda t_0} - T(t_0))v_n = y_n = (\lambda - A)B(\lambda, t_0)v_n = (\lambda - A)u_n \rightarrow y.$$

As $B(\lambda, t_0)$ is invertible and $(\lambda - A)$ is closed then

$$(\lambda - A)u_n \rightarrow (\lambda - A)u = (\lambda - A)B(\lambda, t_0)B^{-1}(\lambda, t_0)u = (\lambda - A)B(\lambda, t_0)h.$$

Then $y_n = (e^{\lambda t_0} - T(t_0))v_n \rightarrow y = (\lambda - A)B(\lambda, t_0)h = (e^{\lambda t_0} - T(t_0))h \Rightarrow y \in R(e^{\lambda t_0} - T(t_0)) \Rightarrow R(e^{\lambda t_0} - T(t_0))$ is closed, consequently the operator $(e^{\lambda t_0} - T(t_0))$ is semiregular.

For the left essential spectrum: Let $t_0 > 0$ be fixed and suppose that $\lambda \notin \sigma_\pi(A)$ for all $\lambda \in \mathbb{C} \setminus \lambda + 2\pi i k t_0^{-1}$. We show that $e^{\lambda t_0} \notin \sigma_\pi(T(t_0))$. By using Lemma(1.1)(3) one has $\dim N(\lambda - A) < \infty$ as one has $T(t)$ is periodic we infer that $\dim N(e^{\lambda t_0} - T(t_0)) = \dim(\overline{\text{span}} \bigcup_{n \in \mathbb{Z}} (\lambda + 2\pi i k t_0^{-1} - A)) < \infty$.

For prove that $R(e^{\lambda t_0} - T(t_0))$ is closed same prove the first party.

For the essential regular spectrum: One suppose that $\lambda - A$ is essentially semiregular for all $\lambda \in \mathbb{C} \setminus \lambda + 2\pi i k t_0^{-1}$. We show that $e^{\lambda t_0} - T(t_0)$ is essentially semiregular.

As $\lambda - A$ is essentially semiregular from (2) of proposition(2.1), it follows that the

operator $\lambda - \hat{A}$ is upper semi-Fredholm where \hat{A} is the generator of the quotient semigroup $(\hat{T}(t_0))_{t \geq 0}$ defined in the first case. Thus, $\lambda \notin \sigma_\pi(\hat{A})$. By virtue of the precedent case we get $e^{\lambda t_0} \notin \sigma_\pi(\hat{T}(t_0))$. Then the operator $(e^{\lambda t_0} - \hat{T}(t_0))$ is upper semi-Fredholm. Next, let $\pi : X \rightarrow X/M$ be the canonical projection. By using Lemma(2.1)(3) with $\dim(N(e^{\lambda t_0} - \hat{T}(t_0))) < \infty$, it can be verified that

$$N(e^{\lambda t_0} - T(t_0)) \subseteq \pi^{-1}(N(e^{\lambda t_0} - \hat{T}(t_0))) \subset M + G = R^\infty(e^{\lambda t_0} - T(t_0)) + G$$

for a finite dimensional subspace G of X .

In fact, we have $N(e^{\lambda t_0} - T(t_0)) \subset N(e^{\lambda t_0} - \hat{T}(t_0))$ and $\pi(N(e^{\lambda t_0} - T(t_0))) \subseteq N(e^{\lambda t_0} - \hat{T}(t_0)) \Rightarrow N(e^{\lambda t_0} - T(t_0)) \subseteq \pi^{-1}N(e^{\lambda t_0} - \hat{T}(t_0)) = \pi^{-1}(N(e^{\lambda t_0} - T(t_0)) + M) = \pi^{-1}(N(e^{\lambda t_0} - T(t_0))) + M \subset G + M$.

The closedness of $R(e^{\lambda t_0} - T(t_0))$ can be proved in exactly the same way as in the first case.

3. B-FREDHOLM SPECTRUM FOR C_0 -SEMIGROUPS

In [14] it is proved the spectral inclusion for semigroups holds for the Fredholm spectrum. In this section we will prove that this result also holds for B-Fredholm spectrum. Recall from [6] that If Y is a closed subspace of X such that $T(t)Y \subseteq Y$ for all $t \geq 0$, that is, if Y is $(T(t))_{t \geq 0}$ -invariant, then the restrictions

$$T(t)|_Y = T(t)|_Y$$

form a strongly continuous semigroup $(T(t)|_Y)_{t \geq 0}$, called the subspace semigroup, on the Banach space Y . The part of A in Y is the operator $A|_Y$ defined by

$$A|_Y y = Ay$$

with domain

$$D(A|_Y) = \{y \in D(A) \cap Y : Ay \in Y\}.$$

In other words, $A|_Y$ is the maximal operator induced by A on Y which coincides with the generator of the semigroup $(T(t)|_Y)_{t \geq 0}$ on Y .

In the sequel, we will prove that the following lemma holds.

Lemma 3.1. *Let A the generator of C_0 semigroup $T(t)_{t \geq 0}$ and $B(\lambda, t) = \int_0^t e^{\lambda(t-s)} T(s) ds$ is a linear bounded operator on X , then there exist two operators C and D such that $(\lambda - A), B(\lambda, t), C, D$ are mutually commuting operators, for all $x \in D(A)$ and $C(\lambda - A)x + DB(\lambda, t)x = Ix$ for $t > 0$.*

Proof: From lemma 2.1 we have:

$$(e^{\lambda t} - T(t))x = B(\lambda, t)(\lambda - A)x, \text{ for } \lambda \in \mathbb{C} \text{ and } x \in D(A).$$

Hence

$$(I - e^{-\lambda t} T(t))x = e^{-\lambda t} B(\lambda, t)(\lambda - A)x, \text{ for } \lambda \in \mathbb{C} \text{ and } x \in D(A).$$

Subsequently :

$$Ix = e^{-\lambda t} B(\lambda, t)(\lambda - A)x + e^{-\lambda t} T(t)x, \text{ for } \lambda \in \mathbb{C} \text{ and } x \in D(A).$$

If we take integration for both sides from 0 to t , we find that:

$$\int_0^t Ix ds = \int_0^t e^{-\lambda s} B(\lambda, s)(\lambda - A)x ds + \int_0^t e^{-\lambda s} T(s)x ds, \lambda \in \mathbb{C}, x \in D(A).$$

Hence,

$$\begin{aligned}
Ix &= G_\lambda(t)(\lambda - A)x + \frac{1}{t} \int_0^t e^{-\lambda s} T(s) x ds \\
&= G_\lambda(t)(\lambda - A)x + \frac{1}{t} e^{-\lambda t} \int_0^t e^{\lambda(t-s)} T(s) x ds \\
&= C(\lambda - A)x + DB(\lambda, t), \quad \lambda \in \mathbb{C}, x \in D(A);
\end{aligned}$$

where

$$C = G_\lambda(t) = \frac{1}{t} \int_0^t e^{-\lambda s} B(\lambda, s) ds \text{ and } D = \frac{1}{t} e^{-\lambda t}.$$

From [12], $(\lambda - A)$, $B(\lambda, t)$, C , D are mutually commuting operators.

Lemma 3.2. *Let $(\lambda - A)$, $B(\lambda, t)$, C and D be mutually commuting operators in $D(A)$ such that $C(\lambda - A) + DB(\lambda, t) = I$ for $t > 0$, then*

- (1) *For every positive integer n there are $C_n, D_n \in D(A)$ such that $(\lambda - A)^n, B^n(\lambda, t)$, C_n, D_n are mutually commuting and*

$$(\lambda - A)^n C_n + B^n(\lambda, t) D_n = I.$$

- (2) *For every positive integer n , we have*

$$R(e^{\lambda t} - T(t))^n = R(\lambda - A)^n \bigcap R(B^n(\lambda, t)) \text{ and } N((e^{\lambda t} - T(t))^n) = N((\lambda - A)^n) + N(B^n(\lambda, t)).$$

Further more,

$$R^\infty(e^{\lambda t} - T(t)) = R^\infty(\lambda - A) \bigcap R^\infty(B(\lambda, t)) \text{ and } N^\infty(e^{\lambda t} - T(t)) = N^\infty(\lambda - A) + N^\infty(B(\lambda, t)).$$

- (3) $N^\infty(\lambda - A) \subset R^\infty(B(\lambda, t))$ and $N^\infty(B(\lambda, t)) \subset R^\infty(\lambda - A)$.

Proof: 1) For $\lambda \in \mathbb{C}$ and $x \in D(A)$, we have that:

$$\begin{aligned}
Ix &= ((\lambda - A)C + DB(\lambda, t))^{2n-1} x \\
&= \sum_{i=0}^{2n-1} \frac{(2n-1)!}{i!(2n-1-i)!} (\lambda - A)^i C^i B^{2n-1-i}(\lambda, t) D^{2n-1-i} x \\
&= (\lambda - A)^n C_n x + B^n(\lambda, t) D_n x.
\end{aligned}$$

This, for some C_n, D_n commuting with $(\lambda - A)^n, B^n(\lambda, t)$.

2) From lemma 2.1 we have: $R(e^{\lambda t} - T(t)) = R((\lambda - A)B(\lambda, t)) \subset R(\lambda - A) \bigcap R(B(\lambda, t))$.

If $y \in R(\lambda - A) \bigcap R(B(\lambda, t))$ with $y = (\lambda - A)x = B(\lambda, t)x'$ for some $x, x' \in D(A)$,

then we set $w = Cx' + Dx$. For this w , we have

$$\begin{aligned}
(\lambda - A)w &= (\lambda - A)Cx' + (\lambda - A)Dx \\
&= (\lambda - A)Cx' + D(\lambda - A)x \\
&= (\lambda - A)Cx' + Dy = (\lambda - A)Cx' + DB(\lambda, t)x' \\
&= x'.
\end{aligned}$$

From (1), we have $B(\lambda, t)(\lambda - A)w = B(\lambda, t)x' = y$. Hence, we conclude that $R(e^{\lambda t} - T(t))^n = R(\lambda - A)^n \bigcap R(B^n(\lambda, t))$ and $R^\infty(e^{\lambda t} - T(t)) = R^\infty(\lambda - A) \bigcap R^\infty(B(\lambda, t))$. Similarly $N(\lambda - A) + R(B(\lambda, t)) \subset N(e^{\lambda t} - T(t)) \bigcap D(A)$. If $x \in N(e^{\lambda t} - T(t)) \bigcap D(A)$, then $(\lambda - A)^n C_n x + B^n(\lambda, t) D_n x = Ix$ where $(\lambda - A)Cx \in N(B(\lambda, t))$ and $B^n(\lambda, t) D_n x \in N(\lambda - A)$.

Thus, $N(e^{\lambda t} - T(t)) \bigcap D(A) = N(\lambda - A) + N(B(\lambda, t))$. From (1), we conclude that $N(e^{\lambda t} - T(t))^n \bigcap D(A^n) = N(\lambda - A)^n + N(B^n(\lambda, t))$ and $N^\infty(e^{\lambda t} - T(t)) =$

$N^\infty(\lambda - A) + N^\infty(B(\lambda, t))$. If $x \in N(\lambda - A)$, then $x = B(\lambda, t)Dx \in R(B(\lambda, t))$, thus $N(\lambda - A) \subset R(B(\lambda, t))$ and by 1), we have $N(\lambda - A)^n \subset R(B^n(\lambda, t))$ for every positive integer n . If $m \geq n$ then $N(\lambda - A)^n \subset N(\lambda - A)^m \subset R(B^m(\lambda, t))$ so that

$$N(\lambda - A)^n \subset R^\infty(B(\lambda, t)) \text{ and } N^\infty(\lambda - A) \subset R^\infty(B(\lambda, t)).$$

The inclusion $R^\infty(B(\lambda, t)) \subset N^\infty(\lambda - A)$ follows from the symmetry.

Theorem 3.1. *For the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$ one has the spectral inclusion*

$$e^{t\sigma_{BF}(A)} \subseteq \sigma_{BF}(T(t)).$$

Proof: Suppose that $(e^{\lambda t_0} - T(t_0))$ is B-Fredholm for some $\lambda \in \mathbb{C} \setminus \{0\}$ and $t_0 > 0$, then there exists a positive integer n such that $M = R(e^{\lambda t} - T(t_0))^n$ is closed and the restricted semigroup $(e^{\lambda t} - T(t_0))|_M$ is Fredholm.

We will show that $(\lambda - A)$ is B-Fredholm. To this end, we show that $R(\lambda - A)^n$ is closed. Let $x \in \overline{R((\lambda - A)^n)}$, then there exist a sequence $u_k \in D(A^n)$ such that $(\lambda - A)^n u_k \rightarrow x$, hence

$$(e^{\lambda t} - T(t_0))^n u_k = B_\lambda(t_0)^n (\lambda - A)^n u_k \rightarrow B_\lambda(t_0)^n x.$$

Since $M = R(e^{\lambda t} - T(t_0))^n$ is closed, then

$$(e^{\lambda t} - T(t_0))^n u_k \rightarrow (e^{\lambda t} - T(t_0))^n u = B_\lambda(t_0)^n (\lambda - A)^n u,$$

for some $u \in D(A^n)$ this implies that

$$x - (\lambda - A)^n u \in N(B_\lambda(t_0)) \subseteq R(\lambda - A)^n$$

so that $x \in R(\lambda - A)^n$, so $R(\lambda - A)^n$ is closed.

Now, let us to show that $(\lambda - A|_{R(\lambda - A)^n})$ is Fredholm. We have that $(e^{\lambda t_0} - T(t_0))|_M$ is Fredholm with generator $(\lambda - A|_{M \cap D(A)})$. Since $(e^{\lambda t_0} - T(t_0))|_M$ is Fredholm, then $\dim N(e^{\lambda t} - T(t_0))|_M < \infty$ implies that $\dim N(e^{\lambda t} - T(t_0)) \cap R(e^{\lambda t} - T(t_0))^n \cap D(A) < \infty$. From [2, lemma 3.2] and from [10, Lemma 8], we have:

$$N(e^{\lambda t_0} - T(t_0)) \cap R(e^{\lambda t_0} - T(t_0))^n \cap D(A) = N(\lambda - A) \cap R(\lambda - A)^n + N(B(\lambda, t_0)) \cap R^n B(\lambda, t_0),$$

$$\text{hence } \dim(N(\lambda - A) \cap R(\lambda - A)^n \cap D(A)) < \dim(N(\lambda - A) \cap R(\lambda - A)^n + N(B(\lambda, t_0)) \cap R^n(B(\lambda, t_0)) \cap D(A))$$

$$= \dim(N(e^{\lambda t_0} - T(t_0)) \cap R(e^{\lambda t_0} - T(t_0))^n \cap D(A)) < \infty.$$

Then $\dim N(\lambda - A|_{R(\lambda - A)^n \cap D(A)}) < \infty$. Moreover from [10, Lemma 4]

$$\dim\left(\frac{R(\lambda - A)^n}{R(\lambda - A)^{n+1}}\right) \leq \max\left(\frac{R(\lambda - A)^n}{R(\lambda - A)^{n+1}}, \frac{R(B^n(\lambda, t_0))}{R(B^{n+1}(\lambda, t_0))}\right) \leq \dim\left(\frac{R(e^{\lambda t_0} - T(t_0))^n}{R(e^{\lambda t_0} - T(t_0))^{n+1}}\right).$$

So $\lambda - A$ is B-Fredholm.

Remark 1. *By the same argument, we can prove that for the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$ one has the spectral inclusion*

$$e^{t\sigma_\nu(A)} \subseteq \sigma_\nu(T(t))$$

where σ_ν is denote the upper semi-B-Fredholm spectrum, lower semi-B-Fredholm.

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